Income and Poverty in a Developing Economy

Amit K Chattopadhyay
SUPA, School of Physics and Astronomy, The University of Edinburgh, Mayfield Road, Edinburgh, EH9 3JZ, UK. and Department of Physics and Astrophysics, University of Delhi, New Delhi 110007, India.

Graeme J Ackland
SUPA, School of Physics and Astronomy, The University of Edinburgh, Mayfield Road, Edinburgh, EH9 3JZ, UK.

Sushanta K Mallick
School of Business and Management, Queen Mary University of London, Mile End Road, London E1 4NS, UK

(Dated: June 2, 2009)

We present a stochastic agent-based model for the distribution of personal incomes in a developing economy. We start with the assumption that incomes are determined both by individual labour and by stochastic effects of trading and investment. The income from personal effort alone is distributed about a mean, while the income from trade, which may be positive or negative, is proportional to the trader's income. These assumptions lead to a Langevin model with multiplicative noise, from which we derive a Fokker-Planck (FP) equation for the income probability density function (IPDF) and its variation in time. We find that high earners have a power-law income distribution while the low income groups have a Levy IPDF. Comparing our analysis with the Indian survey data (obtained from the world bank website) taken over many years we obtain a near-perfect data collapse onto our model's equilibrium IPDF. The theory quantifies the economic notion of "given other things". Using survey data to relate the IPDF to actual food consumption we define a poverty index [1, 2], which is consistent with traditional indices, but independent of an arbitrarily chosen "poverty line" and therefore less susceptible to manipulation.

PACS numbers: 89.65.Gh,02.50.-r

Poverty has been a feature of all human societies throughout time. The underlying cause is the unequal distribution of personal incomes which is an emergent feature of a free economy, invariably resulting in extreme wealth for a few and relative poverty for many.

Since the work of Pareto [3], the distribution of incomes has been known to have a power law tail at the high end [4]. There have been many models of the dynamic process [3, 5, 6, 8, 18, 19] by which a power-law tail can develop for high incomes. Yet in an interlinked economy the low income distribution emerges from the same dynamics as the high income.

Significantly less effort has been applied to study the distribution of low-incomes, but to study poverty this is the critical part of the IPDF. Empirical data shows that low-income distributions are not well described by a Pareto-style power law with a sharp cutoff, as is typically introduced to obtain a normalizable IPDF. Rather than curve-fitting to data, we seek to model the most elementary processes of economic activity, and to find the distribution which emerges.

Such interacting systems are well described using methods in statistical physics [9, 10, 17]. Our basic idea is to represent each individual as an "agent", generating income through personal labour and trade. We describe the income above starvation level, $y_i(t)$, of each agent, i, with a stochastic dynamical equation which describes both labour and trade. It will turn out that trade is the crucial feature for high income groups, while labour is important for lower income groups.

We postulate that the time variation of agent income has the form of a Langevin equation:

$$\frac{dy_i}{dt} = C(t) - My_i + \eta_i(t)y_i \tag{1}$$

Where C represents the rate of increase in income possible from labour, My represents the increasing difficulty to maintain a high income, and $\eta_i(t)$, a random variable with zero mean, represents the stochastic effects of trading. $C(t)$ is a property of the economy as a whole and is slowly varying in time. Possible gains from employment depend on how the economy as a whole is performing. M is a constant which we shall later determine from empirical data. Note that a non-zero mean for the noise term would be equivalent to a smaller value of M, so no assumption is being made about net benefits of trade.

It can be seen that income from labour alone is the same for each person, however the value of trading is proportional to an individual's current wealth. This mix introduces multiplicative noise which is in contrast with previous dynamical approaches [11, 12, 13, 14] in economics, producing anomalous diffusion from the noise itself, not fractional dynamics [14, 26]. Equation 1 does not map on to any well-known physical system, however, there is increasing neurological evidence for such nonlinear risk taking [15, 16].

While it is easy to postulate reasonable-looking intuitive theories for income distribution, there are no known

fundamental laws, and so empirical verification is essential [20]. The largest dataset available for personal income in a developing economy is that collected by the Indian National Sample Survey Organisation (NSSO) covering incomes of millions of people for almost 40 years [7]. The same survey reveals the fraction of income spent on staple food (cereals). Since food is the absolute minimum necessity for survival, we will base our measure of poverty on expenses related to its consumption.

The raw NSSO data comprises income bands ("expenditure classes") of irregular size, from which we generate cumulative income distribution functions (CDFs). Figure 1 shows three typical graphs out of 21 surveys across more than a million households (household size varies from 4-6) between the years 1959-1991. Once scaled to match the mean income for each year, there is a remarkable data collapse. The inset shows that the IPDF emerging from our model is also in excellent approximation to this functional form, as we now discuss.

We assume that the trading decisions are made before their outcome is known, which indicates that we should use Ito calculus: had we assumed mid-term review of trading strategy it would imply Stratonovich calculus, which leads to an equivalent equation with rescaled M. This leads to a Fokker-Planck equation derived from the Langevin model (eq 1).

$$\frac{\partial \hat{f}}{\partial t}(y,t) = \frac{\partial}{\partial y}\left\{\left[(M+2)y - C(t)\right]\hat{f} + y^2\frac{\partial \hat{f}}{\partial y}\right\} \tag{2}$$

In the steady state ($C(t) = C_0$), this would give us the income distribution:

$$\hat{f}(y, t\to\infty) = \frac{{C_0}^{M+1}}{\Gamma(M+1)}\,\frac{\exp(-C_0/y)}{y^{M+2}} \tag{3}$$

Eigenvalue analysis shows that this solution is stable against perturbations. Equation 2 can be analytically solved using Laplace transforms to obtain the full time-dependent solution as a sum of confluent hypergeometric functions $F(a,b,z)$ with time-dependent coefficients:

$$\hat{f}(y,t) = \sum_{n=0}^{n=\infty} \exp(-\omega_n t)\, g_n(y) \tag{4}$$

where $\omega_n = 2\pi n$ and

$$\begin{aligned} g_n(y) &= A_1\left(\frac{c(t)}{y}\right)^{\alpha_-} F(\alpha_-,\beta_-,-\frac{c(t)}{y}) \\ &+ A_2\left(\frac{c(t)}{y}\right)^{\alpha_+} F(\alpha_+,\beta_+,-\frac{c(t)}{y}) \end{aligned} \tag{5}$$

$$\alpha_\pm = \frac{3+M\pm\sqrt{(1+M)^2+4\omega_n}}{2} \tag{6}$$

$$\beta_\pm = 1\pm\sqrt{(1+M)^2+4\omega_n} \tag{7}$$

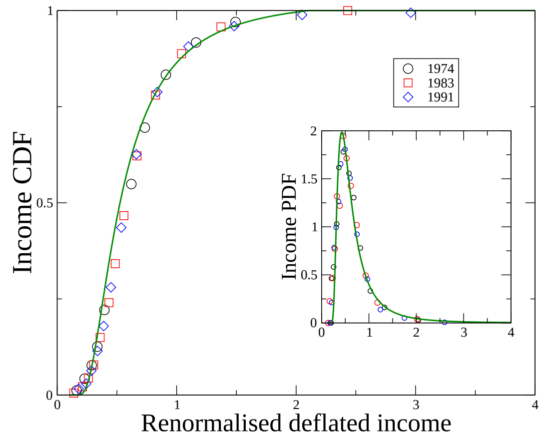

FIG. 1: Plots of the cumulative distribution functions (CDFs) against deflated income for selected years, with inflation independently sourced from the consumer price index (CPI) and renormalized to the 1974 mean income in rupees (64.84 INR). The green line is our theoretical curve, taking y_i as income above a non-zero level below which agents would die of starvation (set at 0.15 in renormalized units). Inset shows the IPDF which is the differential of the CDF, evaluated from the data by interpolation. The points are the data from the NSSO, the lines are our analytic function for the steady state distribution, the only fitting parameters are the power-law tail exponent $M + 2 = 3.6$

and A_1 and A_2 are constants dependent on initial conditions.

Any redistribution measure can be represented by a sum of these functions, and from the associated decay times the timescale of its effect can be determined. However, the data collapse in figure 1 suggests that the relaxation time back to the steady state distribution is rapid.

There is one free parameter, M, which incorporates difficulty of maintaining high income, mean benefit of trade and any possible misconception from our choice of Ito calculus. $M = 1.6$ gives in extremely good agreement with the NSSO data, with $M + 2 = 3.6$ describing the exponent in the power law tail, $C_0\frac{\Gamma(M)}{\Gamma(M+1)}$ being the mean income. This gives us great confidence that our simple, intuitive model does indeed describe the coarse features of the Indian economy, and allows us to proceed with our model of poverty.

Whether poverty-reduction measures are regarded as successful or not often depends on the precise definition of poverty, a semantic which is still argued over. Sen has defined the so-called "axioms of poverty" [1].

- Given other things, a reduction in income of a person below the poverty line must increase poverty.
- Given other things, a pure transfer of income from a person below the poverty line to anyone who is richer must increase poverty.

While these seem to be self-evident, they are based on the ill defined notion of "other things" being unchanged.

This is troublesome: it implies that we should be dealing with partial derivative, but does not specify exactly which variables should be held constant it is not possible simply to increase mean income and "hold everything else constant". Worse, a dynamic system *will* have some non-trivial response to any income reduction of transfer.

By defining the process underlying the income distribution, it becomes possible to define precisely what "given other things" means. For example, increasing mean income corresponds to increasing C in equation 1. A flat-rate tax: increasing low incomes, reducing high incomes, maintaining mean income, would correspond to a multiplier on the $C - My$ term, more complex tax arrangements would change its form altogether. The effect of such changes will induce both a transient response and a steady-state change in IPDF.

In the context of India, there has been considerable debate on the 'true' measure of poverty as the so-called "poverty line" estimates remain controversial [27] yet crucial in the empirical literature on poverty analysis. The official poverty lines which guide Planning Commission policy are based on the nutritional need for calories, but these have been criticized for under-estimating 'true' poverty in India [21].

Three conventional poverty measures involve defining a certain income as the "poverty line", and counting

(i) the fraction of the population with incomes below it (headcount index, HCI)

(ii) The mean percentage below the poverty line (poverty gap index, PG)

(iii) The mean percentage squared below the poverty line (squared poverty gap index, SPG)

A difficulty with such measures is to define the "poverty line", a somewhat arbitrary level of income which also changes with time due to inflation. The successive definitions of poverty measures above reduce the sensitivity of the poverty index to this choice, but do not eliminate it, and pathological cases can easily be derived, especially in practice where NSSO data is discretized into expenditure classes rather than a continuous.

To define a more robust poverty measure, we apply the idea of consumption deprivation (CD) for a specific resource [22, 23, 24]. This uses the fact that expenditure on cereals is monotonically increasing with income, but flattens above a certain income, reflecting the saturation of demand for cereal once one has sufficient to avoid malnutrition.

Correlating the NSSO income data with that for cereal expenditure, we find a good fit to a Monod relationship

$$CD(y) = VK/(K + y) \tag{8}$$

where the parameters V and K are time dependent [23]. Broadly, K can be taken as a "poverty line" which accounts indirectly for cereal-price inflation as opposed to general inflation. V measures the deflated price of cereals. A more intuitive measure of deprivation is the quantity $CD(y)/V$, which is the fraction of the maximum desirable cereal consumption actually consumed by someone of income y.

The advantage of this measure is that it is based on people's actual choices, not on the price of an arbitrarily chosen "basket of goods". So, for example, increasing housing or clothing costs may affect CD even when cereal prices are steady, as income has to be moved from one commodity to another to balance the overall budget. Similarly CD is not affected by changes in the CPI due to price shifts of luxury goods purchased only by the wealthy.

Perhaps most importantly, cereal consumption is directly measured by the NSSO. This allows us to assign a level of poverty to each such NSSO "expenditure classes". By summing this measure, a poverty index based on actual consumption deprivation may be evaluated. We refer to this as the CD-index of poverty, P_{CD}.

Our model allows us to quantify this CD-index of poverty. The model definition of the CD-index satisfies the standard axioms of a poverty index [1, 2], eliminates the arbitrary "poverty line", *and* makes explicit the meaning of "given other things". Using the NSSO data, we can fit an analytic form [23] to the ratio of grain expenditure to income. The CD-index is then defined by the relation

$$P_{CD}(t) = \int \frac{V(t)K(t)}{K(t)+y}\hat{f}(y,t)dy \tag{9}$$

where parameters $V,\ K$ are obtained from NSSO data while $\hat{f}(y,t)$ is the solution of equation 2. The income data used to parameterize our model is independent of the consumption data used to measure CD directly. In Fig. 2, we compare the $P_{CD}(t)$ evaluated directly from the NSSO *consumption* data, and indirectly from our *income-data* based model. We also show the PG and SPG indices. All indices show poverty declining in time, with a peak due to sharp drops in income in the 1960s. However the CD-index shows the effect of increasing cereal prices between 1978-84 as causing an increase in poverty, an effect which cannot be captured in the standard indices.

Against the CPI-deflated data, we see that mean incomes have generally risen over the last forty years, while the relative price of cereals ($V(t)$) has dropped steadily (see Supplementary Material). This helps to reduce poverty, although more direct targetting [25] may be even more effective.

Returning to the stationary IPDF, the power law exponent M is seen to be a crucial component in quantifying the mean income: $C_0\Gamma(M)/\Gamma(M+1)$. Critically, since we have shown that if trading is, on average, beneficial rather than neutral, it will reduce M. Small M increases

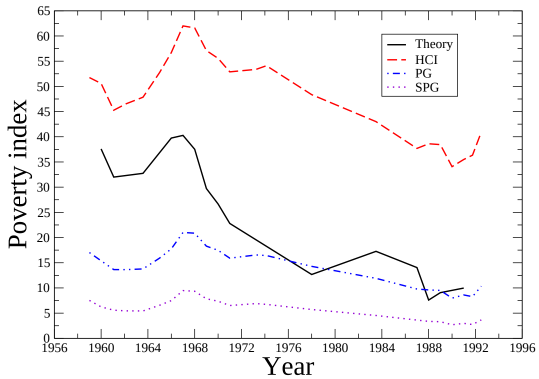

FIG. 2: Plots of the CD-index, a measure of poverty [23], against time. The broken line shows the CD-index as per the official head-count poverty index while the continuous line shows the poverty index arising from our IPDF. To generate the latter we have taken the function $C(t)$ in equation 1 to be a piecewise-linear interpolation of the NSSO-measured mean income at each round and assumed a relaxation time less than a year. Also shown are the poverty gap and squared poverty gap indices, with the poverty line set at the official rate 356 Indian rupees per month. The absolute numbers cannot be compared, but the similarity in trends is evident. $V(t)$ and $K(t)$ are defined from the NSSO data: graphs of V(t), K(t) and C(t) are given in the Supplementary Material.

both the mean income and the level of inequality - it transfers capital from lower to higher income groups.

This illustrates a problem with Sen's axioms. Raising mean incomes "given other things" reduces poverty, while transferring income to higher income groups "given other things" increases poverty. So, in this worldview, the effect of beneficial trade on poverty depends on the definition of "other things". Although one can devise pathological cases, what we find here is that the effect of increasing trade ($M \approx 1$) is to reduce absolute poverty provided the mean income is above the "poverty line" for a headcount index or K for P_{CD}. However, it also has the effect of increasing measures of "relative poverty" where the "poverty line" is a fixed fraction of the mean income.

In summary, we have postulated a stochastic model for the evolution of the income distribution in a developing economy. The steady state of the distribution is stable and robust, and in excellent agreement with the massive NSSO data set for Indian incomes over many years. The existence of an underlying probability distribution function parameterized by mean income makes it much easier to estimate poverty than existing measures such as the head-count index. Under this measure the poverty index is completely specified by the data, without recourse to defining a "poverty line". Moreover, the measure is less susceptible to manipulation by distortions to the income distribution around the poverty line: "lifting people out of poverty" (just). A major strength of this theory lies in its potential power of predicting the response of the IPDF, and hence the poverty index, to external effects, up to a reasonably close (perturbative) time span.

GJA and AKC would like to thank EPSRC for support of the NANIA project and discussions with J.Tailleur. AKC acknowledges helpful discussions with G. Rowlands, University of Warwick and R. Metzler, Technical University of Munich.

[1] A. K. Sen, Econometrica **44**, 219 (1976).
[2] N. C. Kakwani, Econometrica **43**, 437 (1980).
[3] B.B. Mandelbrot, Int. Econom. Rev. **1**, 79 (1960).
[4] R.N. Mantegna and H. E. Stanley, Nature 376, 46 (1995).
[5] E.W. Montroll and M.F. Shlesinger, J. Stat. Phys. **32**, 209 (1983).
[6] J. -P. Bouchaud and M. Potters, in the *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management* (Cambridge University Press, 2003), pp. 87.
[7] National Sample Survey, NSSO, Government of India (www.worldbank.org/poverty/data/indiadata.htm).
[8] R. N. Mantegna, H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance,* Cambridge University Press (Cambridge, 1999).
[9] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Letts. **84**, 1824 (2000).
[10] P. Jefferies, M. L. Hart, P .M. Hui and N.F.Johnson, Euro. Phys. J. B **20,** 493 (2001).
[11] J. -P. Bouchaud and R. Cont, European Physics Journal B **6**, 543 (1998).
[12] H. Takayusu, A. H. Sato and M. Takayusu, Phys. Rev. Letts **79**, 996 (1997).
[13] P. Richmond and L. Sabatelli, Physica A **336**, 27 (2004).
[14] S. Picozzi and B. J. West, Phys. Rev. E **66**, 046118 (2002).
[15] P. W. Glimcher and A. Rustichini, Science **306** 447 (2004).
[16] M. R. Delgado, A. Schotter, E. Y. Ozbay and E. A. Phelps, Science **321** 1849 (2008).
[17] B. During, D. Matthes and G. Toscani, Phys. Rev. E **78**, 056103 (2008).
[18] A. Chatterjee and B. K. Chakrabarti, European Physical Journal B **60**, 135 (2007).
[19] P. K. Mohanty, Phys. Rev. E **74**, 011117 (2006).
[20] J. -P. Bouchaud, Real-world economics review **48** 290 (2008); Nature **455**, 1181 (2008).
[21] R. K. Palmer-Jones, A. K. Sen, J. Development Studies **40** (1), 1 (2003).
[22] M. Pradhan, M. Ravallion, The Review of Economics and Statistics **82** (3), 462 (2000).
[23] T. K. Kumar, A. P. Gore, V. Sitaraman, Journal of Statistical Planning and Inference **49** (1), 53 (1996).
[24] V. Sitaramam, S. A. Paranjpe, T. K. Kumar, A. P. Gore, J. G. Sastry, Economic and Political Weekly, Special Number **31** (35-37), 2499 (1996).
[25] J. Kaiser, Science **319**, 754 (2008).
[26] R. Metzler and J. Klafter, J. Phys. A **37**, 161 (2004).
[27] A. B. Atkinson, Econometrica **55**, 749 (1987).